LIFTING LOCALLY HOMOGENEOUS GEOMETRIC STRUCTURES

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Abstract. We prove that under some purely algebraic conditions every locally homogeneous structure modelled on some homogeneous space is induced by a locally homogeneous structure modelled on a different homogeneous space.

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1. Introduction

The theory of locally homogeneous geometric structures on manifolds is well understood in low dimensions [3]. As the dimensions get higher, researchers notice more prominently the difficulties arising from the complicated equivariant covering maps between different homogeneous spaces. There are two common methods to construct locally homogeneous geometric structures: (1) explicitly write down a discrete subgroup Γ of a Lie group, and an open set U of an associated homogeneous space which is invariant under that discrete subgroup, acted on freely and properly, and take the obvious structure on $\Gamma\backslash U$, or (2) deform such an example via an implicit function theorem argument. The second type of example does not have an explicit description; therefore it is essential to be able to argue about such geometric structures implicitly.

Our aim in this paper is to provide elementary criteria, using only rough data about a locally homogeneous structure, to prove that a structure with one homogeneous model arises from a structure with a different homogeneous model, perhaps on some finite covering space.

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2. Definitions

2.1. **G/H-structures.** Suppose that G is a Lie group and that $H \subset G$ is a closed subgroup.

Definition 1. A *G/H-chart* on a manifold M is a local diffeomorphism from an open subset of M to an open subset of G/H.

Definition 2. Two G/H-charts f_0 and f_1 on a manifold are *compatible* if there is some element $g \in G$ so that $f_1 = gf_0$.

Definition 3. A *G/H-atlas* on a manifold M is a collection of mutually compatible G/H-charts.

Definition 4. A G/H-structure on a manifold M is a maximal G/H-atlas.

2.2. **Pulling back.**

Definition 5. If $F \colon M_0 \to M_1$ is a local diffeomorphism, and $H \subset G$ a closed subgroup of a Lie group, then every G/H-structure on M_1 has a *pullback structure* on M_0, whose charts are precisely the compositions $F \circ f$, for f a chart of the G/H-structure. Conversely, if F is a normal covering map, and M_0 has a G/H-structure which is invariant under all deck transformations, then it induces a G/H-structure on M_1.

2.3. **Developing maps and holonomy morphisms.**

Definition 6. Suppose that (M, m_0) is a pointed manifold, with universal covering space $\left(\tilde{M}, \tilde{m}_0\right)$. Suppose that $H \subset G$ is a closed subgroup of a Lie group.

A *G/H-developing system* is a pair (δ, h) of maps, where

$$\delta \colon \tilde{M} \to G/H$$

is a local diffeomorphism and

$$h \colon \pi_1(M) \to G$$

is a group homomorphism so that

$$\delta(\gamma \tilde{m}) = h(\gamma)\, \delta(\tilde{m}),$$

for every $\gamma \in \pi_1(M)$ and $\tilde{m} \in \tilde{M}$. The map δ is called the *developing map*, and the morphism h is called the *holonomy morphism* of the developing system.

Definition 7. Denote the universal covering map of a pointed manifold (M, m_0) as

$$\pi_M \colon \left(\tilde{M}, \tilde{m}_0\right) \to (M, m_0).$$

Given a G/H-developing system (δ, h) on a manifold M, the *induced G/H-structure* on M is the one whose charts are all maps f so that $\delta = f \circ \pi_M$.

Remark 1. Conversely, it is well known [3] that every G/H-structure is induced by a developing system (δ, h), which is uniquely determined up to conjugacy:

$$(\delta, h) \mapsto (g\delta, \mathrm{Ad}(g)h)$$

and replacing h by any group morphism h' so that $h^{-1}h'$ is valued in the kernel of (G, H) (see section 2.5 on the facing page for the definition of the kernel).

2.4. Inducing structures from other structures.

Definition 8. Suppose that $H_0 \subset G_0$ and $H_1 \subset G_1$ are closed subgroups of Lie groups. A *morphism* of homogeneous spaces (δ, h) is a choice of Lie group morphism $h\colon G_0 \to G_1$ so that $h(H_0) \subset H_1$, and δ is the smooth map

$$\delta\colon g_0 H_0 \in G_0/H_0 \mapsto h(g_0) H_1 \in G_1/H_1.$$

If

$$h'(1)\colon \mathfrak{g}_0/\mathfrak{h}_0 \to \mathfrak{g}_1/\mathfrak{h}_1$$

is a linear isomorphism, then we say that (δ, h) is an *inducing morphism.*

If M is a manifold equipped with a G_0/H_0-chart f, then $\delta \circ f$ is clearly a G_1/H_1-chart. A G_0/H_0-structure $\{f_\alpha\}$ has *induced* G_1/H_1-structure $\{\delta \circ f_\alpha\}$. Every G_0/H_0-developing system (δ_0, h_0) on M has induced G_1/H_1-developing system $(\delta_1, h_1) = (\delta \circ \delta_0, h \circ h_0)$.

2.5. Quotienting by the kernel.

Definition 9. If $H \subset G$ is a closed subgroup of a Lie group, the *kernel* of the pair (G, H) is

$$K = \bigcap_{g \in G} g H g^{-1},$$

i.e. the largest subgroup of H which is normal in G. The kernel is precisely the set of elements of G which act trivially on G/H.

Example 1. If (G, H) has kernel K, we can let $\bar{G} = G/K$, $\bar{H} = H/K$, make the obvious morphism $\bar{}\colon g \in G \mapsto \bar{g} = gK \in \bar{G}$, and then clearly $G/H = \bar{G}/\bar{H}$. Every G/H-structure then has induced $\bar{G}/\bar{H}$-structure, with the same charts, called the *induced effective structure.* Any developing system (δ, h) for the G/H-structure gives the obvious developing system $\left(\bar{\delta}, \bar{h}\right) = (\delta, \bar{} \circ h)$.

2.6. Statement of the theorems.

Theorem 1. *Suppose that $(\delta, h)\colon G_0/H_0 \to G_1/H_1$ is an inducing morphism.*

Uniqueness: Any G_1/H_1-structure on a connected manifold M is induced by at most one G_0/H_0-structure up to multiplying the holonomy morphism by a map $\pi_1(M) \to K_0$ to the kernel K_0 of (G_0, H_0).

Existence: Denote by K_1 the kernel of (G_1, H_1). Suppose that

$$h^{-1} H_1 = H_0 h^{-1} K_1.$$

Pick a G_1/H_1-structure on a connected manifold M with a developing map δ_1 with image in the image of δ, and a holonomy morphism h_1 with image in the image of h. Then the G_1/H_1-structure on M is induced by a G_0/H_0-structure.

Sharpness: if on the other hand

$$h^{-1} H_1 \neq H_0 h^{-1} K_1$$

and $\pi_0\left(h^{-1} H_1 / H_0\right)$ acts trivially on $\pi_0(G_0/H_0)$ (for example if G_0/H_0 is connected), then the standard G_1/H_1-structure on any component of G_1/H_1 is not induced by any G_0/H_0-structure.

Theorem 2. *Suppose that $(\delta, h)\colon G_0/H_0 \to G_1/H_1$ is an inducing morphism and that the kernel of (G_0, H_0) is trivial. Suppose that $h^{-1} K_1 H_0$ has finite index as a subgroup of $h^{-1} H_1$.*

Pick a G_1/H_1-structure on a manifold M. Suppose that the fundamental group of M is finitely generated. Suppose that the developing map of this structure has image in the image of δ, and the holonomy morphism of this structure has image in the image of h. Then there is some finite covering space of M on which the pullback G_1/H_1-structure arises from a G_0/H_0-structure.

On the other hand, suppose that $h^{-1}K_1H_0$ has infinite index as a subgroup of $h^{-1}H_1$. Suppose also that $\pi_0\left(h^{-1}H_1/H_0\right)$ acts on $\pi_0\left(G_0/H_0\right)$ with a finite number of orbits. Then the standard G_1/H_1-structure on any component of G_1/H_1 does not have any finite covering space on which the pullback G_1/H_1-structure arises from any G_0/H_0-structure.

3. Proof of the theorems

Proof of theorem 1 on the previous page:

Proof. Because M is connected, we can throw away all but the component of G_0/H_0 that contains the point $1 \cdot H_0$, and throw away any components of G_0 that take that component to any other component without changing developing map or holonomy morphism. So we can assume that G_0/H_0 and G_1/H_1 are connected.

We can replace G_1 by the image of G_0, and H_1 by its intersection with that image. To show that this does not alter the generality of our proof, we need to show that these replacements won't have much effect on the index of $H_0h^{-1}K_1$ as a subgroup of $h^{-1}H_1$. These replacements have no effect on H_0 or on $h^{-1}H_1$. They change $h^{-1}K_1$ from

$$h^{-1} \bigcap_{g_1 \in G_1} g_1 H_1 g_1^{-1}$$

to the larger group

$$h^{-1} \bigcap_{g_1 \in G_1 \cap h(G_0)} g_1 H_1 g_1^{-1}.$$

Therefore the index of these subgroups after replacement might decrease, but cannot increase.

So we can assume that h and δ are onto. Let $X_0 = G_0/H_0$ and $X_1 = G_1/H_1$ and let $x_0 = 1 \cdot H_0 \in X_0$ and $x_1 = 1 \cdot H_1 \in X_1$ be the identity cosets. Consequently δ is a fiber bundle morphism

$$\begin{tikzcd} X_0 \arrow[d, "\delta"] & \left(h^{-1}H_1\right)/H_0 \arrow[l] \\ X_1, & \end{tikzcd}$$

and so a covering map ([4] p. 121).

Because $\tilde{M}$ is simply connected, the map δ_1 lifts to a unique local diffeomorphism which we naturally denote by

$$\delta_0 \colon \left(\tilde{M}, \tilde{m}_0\right) \to (X_0, x_0).$$

We have only to ask if there is a lift

$$h_0 \colon \pi_1(M) \to G_0$$

of h_1 so that

$$\delta_0 \circ \gamma = h_0(\gamma)\,\delta_0$$

for every $\gamma \in \pi_1(M)$. If we assume that G_0 acts faithfully on G_0/H_0, then there is at most one element of G_0, call it $h_0(\gamma)$, which satisfies this equation. So there is at most one G_0/H_0-structure inducing the given G_1/H_1-structure. If G_0 doesn't act faithfully, then there is at most one such element up to multiplication by an arbitrary element of K_0, since elements of G_0 have the same action as deck transformations on some open set if and only if they agree on all of G_0/H_0, i.e. if and only if they differ by an element of K_0.

We still have to determine when there exists such an element $h_0(\gamma)$. Denote the universal covering map of M by

$$\pi_M \colon \left(\tilde{M}, \tilde{m}_0\right) \to (M, m_0).$$

Let δ_0 be the lift of δ_1 to a continuous map

$$\delta_0 \colon \left(\tilde{M}, \tilde{m}_0\right) \to (X_0, x_0).$$

Given any $\gamma \in \pi_1(M)$, we have an element $h_1(\gamma) \in G_1$ so that

$$\delta_1 \gamma = h_1 \gamma \delta_1.$$

Since h is onto, for any fixed element $\gamma \in \pi_1(M)$, there is some $g_0 \in G_0$ so that $h(g_0) = h_1(\gamma)$. This element g_0 is uniquely determined up to multiplication by an element of $h^{-1}K_1$. But it is possible that $\delta_0 \circ \gamma$ is not equal to $g_0\delta_0$. Since these two maps compose with the covering map δ to the same map to G_1/H_1, they will be equal if and only if they are equal at one point, say at $\tilde{m}_0$. In order that we can arrange them to be equal by choice of g_0, we will need to have some choice of $k_0 \in h^{-1}K_1$ so that $g_0 k_0 H_0 = \delta_0(\gamma\tilde{m}_0)$, i.e.

$$k_0 H_0 = g_0^{-1}\delta_0(\gamma\tilde{m}_0).$$

The point on the right hand side of this equation has the form $g_0' H_0$ where $g_0' \in h^{-1}H_1$. So it suffices that every element of $h^{-1}H_1$ can be written as a product $k_0 h_0$ for some $k_0 \in h^{-1}K_1$ and $h_0 \in H_0$.

Suppose that the kernel of (G_0, H_0) is trivial. Each element γ gives us a unique element g_0 as above, which we write as $h_0(\gamma)$. The map h_0 must be a morphism of groups, by uniqueness of the choice.

Suppose that the kernel of (G_0, H_0) is not trivial, and denote it by K_0. Since h is onto, it is easy to check that $h(K_0) \subset K_1$. Consider the induced effective structures. Define the obvious morphism

$$\bar{h} \colon \bar{G}_0 \to \bar{G}_1$$

and map

$$\bar{\delta} \colon \bar{G}_0/\bar{H}_0 \to \bar{G}_1/\bar{H}_1.$$

We leave the reader to check that

$$\bar{h}^{-1}\bar{H}_1 = \bar{H}_0\bar{h}^{-1}\bar{K}_1.$$

We then apply the above results to ensure that we can lift the induced effective structure, i.e. the $\bar{G}_1/\bar{H}_1$-structure, to a unique $\bar{G}_0/\bar{H}_0$-structure. We then define a G_0/H_0-structure by taking as charts precisely the same maps

$$\text{open subset } \subset M \to \bar{G}_0/\bar{H}_0 = G_0/H_0.$$

Via δ this induces the original G_1/H_1-structure, because the set of charts induces the $\bar{G}_1/\bar{H}_1$-structure, which has the same charts as the G_1/H_1-structure.

Suppose finally that $H_0 h^{-1} K_1 \neq h^{-1}H_1$. Let $M \subset X_1$ be the component of x_1, with the standard G_1/H_1-structure, developing map

$$\delta_1 \colon \tilde{M} \to X_1$$

the universal covering map and holonomy morphism

$$h_1 \colon \pi_1(M) \to G_1$$

given by $h_1(\gamma) = 1$ for all $\gamma \in \pi_1(M)$. Suppose that we found a lift to a G_0/H_0-structure, with developing map

$$\delta_0 \colon \tilde{M} \to X_0$$

and holonomy morphism

$$h_0\colon \pi_1(X_1) \to G_0,$$

lifting h_1. So $h \circ h_0 = h_1 = 1$, and therefore h_0 is valued in $\ker h$.

The inclusion $H_0 h^{-1} K_1 \subset h^{-1} H_1$ is clear. Take any element $g_0 \in h^{-1} H_1$. We need to prove that $g_0 \in H_0 h^{-1} K_1$. Take any path $x(t) \in X_0$ from $x_0 = 1 \cdot H_0$ to $g_0 x_0 = g_0 \cdot H_0$. Let γ be the homotopy class of the loop $\delta \circ x(t)$ in M. Then

$$\begin{aligned} \delta_0(\gamma \tilde{m}_0) &= g_0 H_0 \\ &= h_0(\gamma)\,\delta_0(\tilde{m}_0) \\ &= h_0(\gamma)\,H_0. \end{aligned}$$

So $g_0 \in h_0(\gamma) H_0$, and therefore $g_0 \in h^{-1} K_1 H_0$. But $h^{-1} K_1 H_0 = H_0 h^{-1} K_1$ since $K_1 \subset G_1$ is a normal subgroup. □

Example 2. Suppose that $(\delta, h)\colon G_0/H_0 \to G_1/H_1$ is an inducing morphism and that $H_1 \subset G_1$ is a normal subgroup. (For example, if G_1 is abelian.) Then $K_1 = H_1$ so $h^{-1}H_1 = H_0 h^{-1} K_1$. Therefore every G_1/H_1-structure on any manifold lifts to a G_0/H_0-structure.

Example 3. Let $G_0 = \mathrm{SL}(2,\mathbb{C})$, H_0 the stabilizer of some complex line in $\mathbb{C}^2$, $G_1 = \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$, H_1 the stabilizer of the point in $\mathbb{P}^1$ associated to that complex line. Take the obvious morphism $h\colon G_0 \to G_1$, so that $h^{-1}H_1 = H_0$. Then $K_0 = \pm I, K_1 = I$ and $h^{-1}H_1 = H_0 = H_0 h^{-1} K_1$, so every holomorphic projective structure on any Riemann surface lifts to an $\mathrm{SL}(2,\mathbb{C})/H_0$-structure. Moreover, the possible lifts h_0 are determined up to changing by a map to $K_0 = \pm I$. On a given surface M, pick any generators for $\pi_1(M)$ and we can change h_0 arbitrarily by multiplying by $\pm I$ on each these generators. So on a compact Riemann surface of genus g, there are precisely 2^{2g} lifts. This result was previously proven using techniques which are very specific to holomorphic projective structures; [2] lemma 1.3.1 p. 632

Example 4. The same argument works for $\mathrm{SL}(n+1,\mathbb{C})$ instead of $\mathrm{SL}(2,\mathbb{C})$. If the fundamental group of a manifold with complex projective structure is finitely generated, then there are $b_1(M)^{n+1}$ different lifts.

Example 5. More generally still: suppose that G_0 is a complex semisimple Lie group in its simply connected form, and $H_0 \subset G_0$ is a parabolic subgroup. Suppose that G_1 is the same complex semisimple Lie group in its adjoint form, and $H_1 \subset G_1$ the corresponding parabolic subgroup. Then $K_1 = \{1\}$, and $h^{-1}H_1 = H_0$, while $K_0 = \pi_1(G_1) = Z(G_0)$ is the center of G_0, so every G_1/H_1-structure (i.e. flat holomorphic parabolic geometry) lifts to a G_0/H_0-structure. On a manifold M with finitely generated fundamental group, the number of different lifts is $b_1(M)^{|Z(G_0)|}$.

Example 6. We give another example where the map $G_0/H_0 \to G_1/H_1$ is an infinite covering map. Our example will be complex analytic. The group $G_0 = \mathrm{GL}(2,\mathbb{C})$ acts transitively on $G_0/H_0 = \mathbb{C}^2$ with stabilizer H_0 consisting precisely in the matrices of the form

$$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}$$

for $p \in \mathbb{C}$ and $q \in \mathbb{C}^\times$. Fix a complex number λ with $|\lambda| > 1$, and an integer $n > 0$. Consider the subgroup $Z_0 \subset G_0$ consisting of the matrices of the form

$$\mu \lambda^k I,$$

where μ is any solution of $\mu^n = 1$ and k is any integer.

The group $G_1 = G_0/Z_0$ acts transitively on the smooth compact complex surface

$$S = Z_0 \backslash \left(\mathbb{C}^2 \setminus 0\right),$$

which is a homogeneous Hopf surface [5]. (All homogeneous Hopf surfaces arise by this construction.) The stabilizer of a point of S is the group H_1 which is the quotient modulo Z_0 of all matrices of the form

$$\begin{pmatrix} \mu\lambda^k & p \\ 0 & q \end{pmatrix}.$$

So $h^{-1}H_1$ is precisely this collection of matrices. The kernel K_1 of (G_1, H_1) is trivial, so $h^{-1}K_1$ is the kernel of h, i.e. precisely Z_0. Since clearly every element of $h^{-1}H_1$ has the form of a product of an element of Z_0 with one of H_0, every G_1/H_1-structure, i.e. homogeneous Hopf structure, arises from a unique G_0/H_0-structure. It is known that a compact complex surface bears a G_0/H_0-structure just when it is either (1) a linear Hopf surface or (2) a holomorphic elliptic fibration with no singular fibers and not covered by a product of curves; [1]. Therefore it follows that those are precisely the compact complex surfaces which admit homogeneous Hopf structures.

Example 7. More generally, if $H_0 \subset G_0$ is a closed subgroup of a Lie group, and $Z_0 \subset G_0$ is closed normal subgroup, we can let $G_1 = G_0/Z_0$, and $H_1 = (Z_0H_0)/Z_0$. Then $h^{-1}H_1 = Z_0H_0$ while $h^{-1}K_1$ is the kernel of (G_0, Z_0H_0), which contains Z_0. If the identity component of Z_0 lies in H_0 (for example if Z_0 is discrete), then

$$\mathfrak{g}_0/\mathfrak{h}_0 = \mathfrak{g}_1/\mathfrak{h}_1,$$

so any G_1/H_1-structure lifts to a G_0/H_0-structure.

Example 8. Let G_0 be the orthogonal group in 3 variables, and H_0 the stabilizer of the north pole on the unit sphere, so G_0/H_0 is the unit sphere. Let $Z_0 = \pm 1$, $G_1 = G_0/Z_0$, and $H_1 = Z_0H_0/Z_0$. Then $G_1/H_1 = \mathbb{RP}^2$ is the real projective plane equipped with its standard round Riemannian metric. So every G_1/H_1-structure lifts to a G_0/H_0-structure. This is just saying that a Riemannian metric locally isometric to $\mathbb{RP}^2$ is locally isometric to S^2.

Example 9. Lifting is not always possible. Consider the group $G_0 = \mathrm{SO}(3)$ of rotation matrices in 3 variables, acting on the unit sphere, so H_0 is the collection of rotations fixing the north pole of the sphere. As a homogeneous space, G_0/H_0 is the unit sphere with its usual round metric and orientation. Let $G_1 = G_0$ but H_1 be the set of all rotation matrices which preserve the north-south axis; i.e. either rotate fixing the north pole, or rotate the north pole to the south pole. Then $G_1/H_1 = \mathbb{RP}^2$ with the usual flat Riemannian metric. The kernels of both homogeneous spaces are trivial. The group $h^{-1}H_1$ is just H_1, which is strictly larger than H_0. The group $h^{-1}K_1$ is trivial. Therefore our theorem does not apply. In fact, the standard G_1/H_1-geometry on $\mathbb{RP}^2$ does not arise from any G_0/H_0-geometry on $\mathbb{RP}^2$, because a G_0/H_0-geometry imposes an orientation, and $\mathbb{RP}^2$ is not orientable. This example motivates our search for a criterion for lifting a G_1/H_1-structure to a G_0/H_0-structure on a finite covering space.

Proof of theorem 2 on page 3:

Proof. It is easy to check that $h^{-1}K_1H_0$ is always a subgroup of $h^{-1}H_0$. We need only construct a holonomy morphism as in the proof of theorem 1, for a sufficiently high power of each element of the fundamental group. Because M has finitely generated fundamental group, this will then ensure that a holonomy morphism is defined on a finite index subgroup, so on a finite covering space. So repeating the proof of theorem 1, it suffices that every element of $h^{-1}H_1$ has some power which can be written as a product k_0h_0 for some $k_0 \in h^{-1}K_1$ and $h_0 \in H_0$. Since

$$h'(1) \colon \mathfrak{g}_0/\mathfrak{h}_0 \to \mathfrak{g}_1/\mathfrak{h}_1$$

is a linear isomorphism, and clearly also

$$h'(1) \colon \mathfrak{g}_0/h'(1)^{-1}\mathfrak{h}_1 \to \mathfrak{g}_1/\mathfrak{h}_1$$

is a linear isomorphism,

$$h'(1)^{-1}\mathfrak{h}_1 = \mathfrak{h}_0.$$

Therefore the Lie group H_0 has the same identity component as $h^{-1}H_1$. Since the index of H_0 in $h^{-1}H_1$ is finite, every element of $h^{-1}H_1$ has a finite power in H_0.

There remains only one detail: in the process of the previous proof, we made use of the induced effective structures. Write the kernel of (G_0, H_0) as K_0, etc. as before. We leave the reader to check that the index of $\bar{H}_0\bar{h}^{-1}\bar{K}_1$ as a subgroup of $\bar{h}^{-1}\bar{H}_1$ is unchanged if we drop all of the bars, i.e. the same as the index of $H_0h^{-1}K_1$ as a subgroup of $h^{-1}H_1$.

Suppose that $h^{-1}K_1H_0$ has infinite index as a subgroup of $h^{-1}H_1$. Let M be the component of G_1/H_1 containing the identity coset. The developing map

$$\delta_1 \colon \tilde{M} \to G_1/H_1$$

is the universal covering map. The holonomy morphism

$$h_1 \colon \pi_1(M) \to G_1$$

is $h_1(\gamma) = 1$ for all $\gamma \in \pi_1(M)$. Suppose that $M' \to M$ is a finite covering map, and that the pullback G_1/H_1-structure is induced, say by a developing map

$$\delta_0 \colon \tilde{M} \to G_0/H_0$$

and holonomy morphism

$$h_0 \colon \pi_1(M) \to \ker h.$$

Take any element $g_0 \in h^{-1}H_1$. We need to prove that some power g_0^n lies in $H_0h^{-1}K_1$ for some integer $n > 0$. Replacing g_0 by a suitable g_0^n, for some $n > 0$, we can arrange that g_0x_0 lies in the same path component as x_0 in X_0. Take any path $x(t) \in X_0$ from x_0 to g_0x_0. Let γ be the homotopy class of the loop $\delta \circ x(t)$ in M. Then

$$\begin{aligned} \delta_0(\gamma\tilde{m}_0) &= g_0H_0 \\ &= h_0(\gamma)\,\delta_0(\tilde{m}_0) \\ &= h_0(\gamma)\,H_0. \end{aligned}$$

So $g_0 \in h_0(\gamma)\,H_0$, and therefore $g_0 \in h^{-1}K_1H_0$. But $h^{-1}K_1H_0 = H_0h^{-1}K_1$ since $K_1 \subset G_1$ is a normal subgroup. □

Example 10. Suppose that G_0 is a Lie group and $\Gamma \subset G_0$ is a discrete subgroup. Let $H_0 = \{1\}$, $G_1 = G_0$ and $H_1 = \Gamma$. So we are asking if we can lift any G_0/Γ-structure to a $G_0/\{1\}$-structure. Since K_1 is the largest subgroup of Γ which is normal in G_0, we can lift G_0/Γ-structures to $G_0/1$-structures, possibly by taking a finite cover, as long as Γ has as finite index subgroup normal in G_0.

Example 11. Let G_0 be the group of affine transformations of $\mathbb{R}$, $x \mapsto ax + b$. Represent these as matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Let $H_0 = \{1\}$. Let $G_1 = G_0$. Take $H_1 \subset G_1$ a discrete group. By definition, K_1 is the largest normal subgroup of G_1 lying in H_1. Discrete normal subgroups of a Lie group must lie in the center. But the center of $G_1 = G_0$ is $\{1\}$, so $K_1 = \{1\}$. So

$h^{-1}H_1 = H_1$, while $H_0 h^{-1} K_1 = \{1\}$. Therefore our theorems do not apply unless H_1 is finite, i.e. H_1 is the group of matrices of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or $H_1 = \{1\}$.

Consider for example taking H_1 to be the set of matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for $n \in \mathbb{Z}$. The quotient G_1/H_1 is identified with matrices

$$\begin{pmatrix} a & ab \\ 0 & 1 \end{pmatrix}$$

up to $b \sim b+1$, so $X_1 = \mathbb{C}^{\times} \times (\mathbb{C}/\mathbb{Z})$. The deck transformation

$$\begin{pmatrix} a & ab \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & a(b+n) \\ 0 & 1 \end{pmatrix}$$

is not carried out by any element of G_0, and therefore the G_1/H_1-structure on X_1 does not lift to a G_0/H_0-structure on any finite covering space.

References

1. Sorin Dumitrescu and Benjamin McKay, *Locally homogeneous geometric structures on compact complex surfaces*, to appear.
2. Daniel Gallo, Michael Kapovich, and Albert Marden, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Ann. of Math. (2) **151** (2000), no. 2, 625–704. MR 1765706 (2002j:57029)
3. William M. Goldman, *Locally homogeneous geometric manifolds*, Proceedings of the International Congress of Mathematicians (Hyderabad) (Rajendra Bhatia, ed.), vol. 2, International Congress of Mathematicians, Hindawi, 2010, pp. 717–744.
4. V. V. Gorbatsevich, A. L. Onishchik, and E. B. Vinberg, *Foundations of Lie theory and Lie transformation groups*, Springer-Verlag, Berlin, 1997, Translated from the Russian by A. Kozlowski, Reprint of the 1993 translation [ıt Lie groups and Lie algebras. I, Encyclopaedia Math. Sci., 20, Springer, Berlin, 1993; MR1306737 (95f:22001)]. MR 1631937 (99c:22009)
5. Alan T. Huckleberry, *The classification of homogeneous surfaces*, Exposition. Math. **4** (1986), no. 4, 289–334.

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